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LETTER TO THE EDITOR

Mean field theory of dilute spin-glasses with power-law interactions

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Abstract. We argue that in order to describe dilute dipolar or RKKY spin-glasses, a broad distribution of couplings (i.e. decaying as a power law) is more appropriate than the traditional Gaussian distribution. We present a mean-field 'replica symmetric' theory for this problem. We find that the spin-glass transition is of a different nature than the one occurring in the SK model, and resembles a percolation transition. The de Almeida–Thouless temperature is lower than the spin-glass transition temperature. The degree of 'replica symmetry breaking', measured by the absolute value of the negative entropy, is found to be smaller than in the SK model *and non-monotonic with temperature.* Some experimental consequences are briefly addressed.

Despite many years of effort, the correct theoretical picture of finite-dimensional spin-glasses is still lacking [1, 2]. From the celebrated Parisi solution of the (mean field) Sherrington–Kirkpatrick model, one induces that a spin-glass is characterized by a large number of nearly degenerate ground states, separated by infinite barriers and organized in a hierarchical manner in phase space. From this large degeneracy follows the existence of a transition (de Almeida–Thouless) line in the temperature–magnetic field plane [3], below which ergodicity is broken and anomalous dynamics is expected: in many respects, the system remains 'critical' in the whole spin-glass phase [4–6].

A rather different (although possibly complementary) point of view is expressed by Fisher and Huse [7, 8]: a finite dimensional spin-glass would possess a unique ground state (up to a global spin-flip) above which excitations organize as droplets of reversed spin. The energy of a droplet grows with its linear size as σL^{θ} with $\theta \simeq 0.2$ in $d=3$ [8]. σ depends on the disorder and its probability density is expected to be finite around $\sigma=0$ [7].

Although many qualitative predictions are not so different in the two approaches (although dynamical 'T-jump' experiments may be able to distinguish between the two [9, 10]), the droplet model leads to two rather important conclusions [7, 8]: first, the overlap distribution function should reduce, for large sizes, to a delta function (since there is only one state, but see also [7]), and no de Almeida–Thouless (AT) line should

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exist (the broken symmetry at the spin-glass transition would be a disguised Ising symmetry, rather than the 'replica symmetry'). Recent numerical simulations, however, suggest that some features of the droplet picture should probably be revised [11-13] (but see [14]). $1/d$ expansion 'around' the SK model show furthermore that the distinctive features of Parisi's solution are enhanced, rather than suppressed, when the dimension is lowered [15].

In order to bring a little more to the understanding of the physics of 'real' spin-glasses, we have generalized the SK model to the case of a 'broad' distribution of the couplings J_{ij}, i.e. the case where the variance of J is infinite. Apart from the interest of exhibiting a new family of solvable spin-glasses, the physical motivations are the following:

If one throws at random in space spins which interact through RKKY or dipolar interactions, the resulting distribution of coupling $\mathcal{P}(J)$ decays, for large J, as J^{-2}. This tail corresponds to nearby spins interacting as g/r^3, and is, of course, eventually cut-off by the hard-core interaction or the Fermi wavelength. This however means that there are two important energy scales: the typical interaction strength $g\rho$, which is set by the density of spins ρ, and its root mean square $(\overline{J^2})^{1/2} \sim g/a^3$, which is dominated by the cut-off distance a. For sufficiently small concentrations of spins ($\rho a^3 \ll 1$), the fraction of spins at close contact is negligible, and the range over which the interaction varies is large: it is therefore a very bad approximation to portray $\mathcal{P}(J)$ as a Gaussian. We shall thus consider, more generally, distributions decaying as $\mathcal{P}(J) \sim J^{-(1+\mu)}$ (corresponding to $J(r) \simeq r^{-d/\mu}$), with $\mu \leqslant 2$, for which the RMS is formally infinite ($\mu = 1$ corresponding to dipolar/RKKY interactions, see also [16, 17]). Couplings drawn from such distributions are well separated in scale: for example, the ratio of the largest to the next largest tends [as the number of couplings goes to infinity] to $2^{1/\mu}$ which becomes very large as $\mu \to 0$. Intuitively, one might expect that frustration, which is at the heart of the spin-glass folklore, is diminished as μ decreases: strong bonds are satisfied first, leaving only much weaker bonds unsatisfied. One should thus expect that 'replica symmetry breaking' effects should somehow be reduced, or even possibly disappear, as μ decreases from its Gaussian limit $\mu = 2^+$. In other words, this work intends to discuss the generality of replica symmetry breaking.

Our motivation was also spurred by surprising experimental results on dilute dipolar spin-glasses [18], which showed a very clear tendency for the relaxation time spectrum to *narrow* as the temperature is *decreased*; as we shall argue later, this might be related to some novel behaviour which arises in our model. This anomalous behaviour disappears for higher spin concentrations, where the Gaussian model becomes reasonable.

We consider N Ising spins on a fully connected lattice, with a Hamiltonian given by

$$\mathcal{H} = -\sum_{i<j} \frac{J_{ij}}{N^{1/\mu}} S_i S_j \tag{1}$$

where the J_{ij} are distributed according to a symmetric distribution $\mathcal{P}(J)$, decaying for large $|J|$ as $J_0^{\mu}/|J|^{(1+\mu)}$. As the sum of N such variables scale as $N^{1/\mu}$ (see e.g. [19]), we have rescaled the couplings in (1) so that the energy per spin is of order 1. We study the problem through the 'cavity method' (we have not been able to make progress with the replica method), which amounts to write self-consistent equations when an $(N+1)$th spin S_0 is added to the system. The basic hypothesis is that the probability $P(\{S\})$ to observe $\{S_1, S_2, \ldots, S_N\}$ is factorized—this assumes that the system can be prepared in a single 'pure' phase (in the case of the pure ferromagnet, one may show that this assumption fails only right at the transition point):

$$P(\{S\})=\prod_{i=1}^{N}\left[\frac{e^{\lambda_i S_i}}{2\cosh(\lambda_i)}\right] \tag{2}$$

where $m_i=\tanh(\lambda_i)$ is the magnetization of spin i in the N spin system. From (2) and $h=\Sigma_{i=1}^{N}\hat{J}_{0i}S_i$ ($\hat{J}\equiv J/N^{1/\mu}$), one may obtain the probability distribution of h, $p(h)$. Finally, assuming that the addition of the $(N+1)$th spin only weakly perturbs the $\{m_i\}$, one may calculate the free energy and the magnetization of the extra spin. One finds:

$$Z\equiv\sum_{S_0}\int \mathrm{d}h\, p(h)\, e^{\beta h S_0}=2\, e^{\varphi}\cosh(\Phi) \tag{3}$$

and

$$m_0\equiv Z^{-1}\sum_{S_0}\int \mathrm{d}h\, p(h) S_0\, e^{\beta h S_0}=\tanh(\Phi) \tag{4}$$

with

$$\Phi=\sum_{i=1}^{N}\tanh^{-1}(m_i\tanh(\beta\hat{J}_{0i})) \tag{5}$$

and

$$\varphi=\sum_{i=1}^{N}\log(1+(1-m_i^2)\sinh^2(\beta\hat{J}_{0i})) \tag{6}$$

and $\beta\equiv 1/T$.

Assuming that $\beta\hat{J}_{0i}$ is always small, one finds the familiar equation $m_0=\tanh(\Sigma_{i=1}^{N}\beta\hat{J}_{0i}m_i)$ which holds in the SK model [1]. Naively, one could think that this is always justified due to the $N^{-1/\mu}$ rescaling factor. Of course, this is not true in our case, since the largest (and most important) J is precisely of order $N^{1/\mu}$. It is however easy to see that the usual central limit theorem may be applied to the sum defining Φ, which thus has a Gaussian distribution $G_{\beta^2 q}(\Phi)$ of variance$\equiv\beta^2 q$. The self consistent equation giving q (which generalizes the Edwards–Anderson order parameter) then reads:

$$q=\int_{-\infty}^{+\infty}\mathrm{d}\Phi\, G_{q/T^2}(\Phi)Q_\mu(\tanh(\Phi)) \tag{7}$$

with

$$Q_\mu(y)=2T^2\left(\frac{J_0}{T}\right)^{\mu}\int_0^{+\infty}\frac{\mathrm{d}\varepsilon}{\varepsilon^{1+\mu}}\tanh^{-1}(y\tanh(\varepsilon)).$$

The free energy can be obtained using equation (3) (see [1, 20] for details). In order to check the validity of this calculation (which is similar to the 'replica symmetric' approach to the SK model), one now adds *two* spins and compute their correlations, which must remain of order $1/\sqrt{N}$. This leads to a condition which generalizes the de Almeida–Thouless line to our case [1, 20].

One should finally mention that a mean-field calculation for r^{-3} interacting spins was presented in [16, 21]. The authors use a somewhat different language, but their results amount to the above mentioned expansion in $\beta\hat{J}_{0i}$. This leads to an infinite transition temperature (i.e. governed by the short-scale cut-off), and not, as we shall show below, to a transition at a finite $T \simeq J_0$ ($= g\rho$ in the physical case referred to above).

We now consider the results.

(a) Transition temperature. From equation (7), one finds that there exists a *finite* transition temperature $T_c \equiv J_0(\int_{-\infty}^{+\infty} d\varepsilon/|\varepsilon|^{1+\mu} \tanh^2(\varepsilon))^{1/\mu}$ for all $0<\mu<2$, below which q takes non-zero values, signalling the appearance of frozen spins. One finds $q(T) \propto T_c - T$ for T close to T_c and $q(T) = q(0) - \alpha T^{3-\mu}$ at low temperature. The 'spin-glass' transition is however of a different nature than the one found in the SK model. In the SK model, the transition is associated with a *collective* freezing of all the spins, whereas in our case, the transition is akin to a percolation transition: at T_c an infinite cluster of strongly correlated spins start to freeze, while many finite clusters of spins remain in their high temperature state. As temperature is decreased, these finite clusters progressively merge with the infinite cluster (this scenario was in fact proposed in early spin-glass theories [22]). A way to understand this is to estimate the probability that a given coupling exceeds, say, $2T$: one finds

$$P(|J|>2N^{1/\mu}T) = 2\int_{2N^{1/\mu}T}^{\infty} dJ \frac{J_0^\mu}{J^{1+\mu}} = \frac{2^{1-\mu}}{\mu N}\left(\frac{J_0}{T}\right)^\mu .$$

Note that these 'strong bonds' are not only larger than T but also larger than the sum of all other couplings emerging from the site. One can show (see e.g. [23]) that these 'strong bonds' percolate when there is on average one strong bond per site, i.e. for a temperature such that: $N(2^{1-\mu}/\mu N)(J_0/T)^\mu = 1$ which is indeed of the same order of magnitude as T_c. A very important qualitative conclusion is that there should exist a certain fraction f of 'fast spins' corresponding to finite clusters still in the paramagnetic state, in the spin-glass phase, where, in analogy with percolation, $1-f \simeq (T_c - T)^\beta$ ($\beta \simeq 0.5$ for 3D percolation). These 'fast spins' are indeed very clearly seen in magnetization relaxation experiments [24]: after the field is cut, there is always a very fast initial drop of the magnetization which could be attributed to these 'fast spins'. The relative amplitude of the drop is furthermore found to be roughly compatible with the percolation estimate [24].

(b) Entropy and de Almeida–Thouless instability. A plot of the calculated entropy as a function of reduced temperature T/T_c is given in figure 1 for different values of μ. As in the SK model, the entropy is seen to become negative below a certain

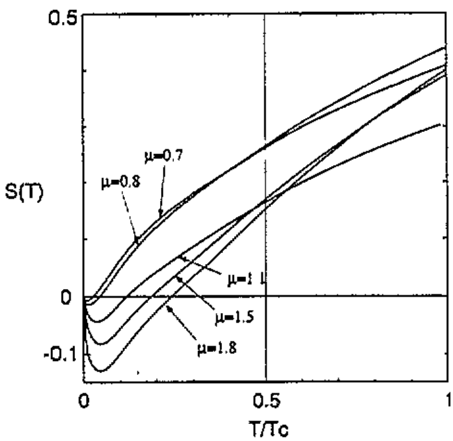

Figure 1. Entropy of the replica symmetric solution in the low-temperature phase, versus reduced temperature T/T_c, for different values of μ. Note that $S(T)$ reaches a minimum for $T>0$, and vanishes for $T=0$. Note that the AT temperature is, in these units, around 0.8–0.85 for all μ.

temperature T^*. Quite surprisingly, however, the entropy reaches a minimum and then increases again to reach zero for $T=0$: replica symmetry breaking, at variance with the SK model, seems to be maximum at a non-zero temperature. One should also notice that the minimum value of the entropy is significantly lower than in the SK model (for example, $S_{\min} \simeq -0.01, -0.08, -0.17$ for $\mu=0.7$, 1.5 and SK).

Near T_c, one can show analytically that the replica symmetric solution is *stable*. For $\mu=1.5$, for example, $T_c \simeq 2.8$, while the AT instability occurs at $T_{AT} \simeq 2.4$. (The ratio T_{AT}/T_c is found to depend very weakly on μ.) Hence we find *two* different transition temperatures, first towards a 'simple' spin-glass phase where freezing of the spins occur, followed (as temperature is decreased) by a 'multi-phase' spin-glass (i.e. replica symmetry broken). This may be qualitatively understood using the percolation picture: just below T_c, the infinite 'cluster' contains very few 'loops' which are responsible for frustration. As the temperature is lowered, the clusters—which can be thought of as 'superspins'—start to interact strongly, and the freezing of the orientation of these superspins leads to frustration and to the AT instability. Experimentally, this raises the intriguing possibility that some freezing might occur before any irreversibility line is crossed.

(c) TAP equations and structure of the ground state. From the cavity equations, one may obtain TAP-like equations which relate the average magnetization on a given site to the average field acting on that site. One finds:

$$m_0^{N+1} = \tanh T^{-1}\left[\sum_{i=1}^{N} \hat{J}_{0i} m_i^{N+1} + \Psi(T) - \frac{\partial \varphi}{\partial(1/T)} m_0^{N+1}\right] \tag{8}$$

where m^{N+1} refers to a thermal average in the $N+1$ system and $\Psi(T) \equiv \Phi(T) - \beta(\partial\Phi(T)/\partial\beta)$. It is easy to check (using (6)) that equation (8) reduces to the usual TAP equation when $\beta\hat{J}_{0i} \ll 1$. At this stage, and for the following discussion, two cases must be distinguished:

(i) $1<\mu<2$: In this case, the average of $|J|$ is finite, as well as the average free energy per spin. However, the additional term in equation (8) does not vanish for $T=0$ (as it should to reproduce the correct equation for the $T=0$ metastable states, $m_0^{N+1} = \mathrm{sgn}[\Sigma_{i=1}^{N} J_{0i} m_i^{N+1}]$). This is because, as in the SK model, the local field density $p(\Phi)$ is non-zero around $\Phi=0$, which leads to $1-m^2 \propto_{T\to 0} T$ for our one-phase solution. A more detailed analysis of equation (8), or a generalization of an argument due to Anderson [25], suggests that true ground states should be such that $p(\Phi) \simeq_{\Phi\to 0} \Phi^\zeta$ with $\zeta \geqslant \mu-1$, while numerical simulations [20] suggest that $\zeta=\mu-1$. Hence the one-phase solution cannot be the correct solution for $T=0$, even if its entropy is zero. From the above result on $p(\Phi)$, we predict that $S(T) \simeq_{T\to 0} T^\mu$.

Finally, we have calculated numerically the number $\mathcal{N}$ of $T=0$ metastable states and their energy distribution. We found that $\log \mathcal{N} \simeq \gamma N$ (for N up to 25), with, surprisingly, $\gamma \sim 0.15$ independently of μ (we recall that $\gamma=0.2$ in the SK model). However the energy distribution is very broad (contrary to the SK case), indicating that usual algorithms to find the ground state energy [4] become quickly inefficient as N increases. For $\mu=1.5$, we have found a ground state energy of $\simeq -2.85J_0$ for $N=100$ and $\simeq -2.7J_0$ for $N=400$, while the above one-phase calculation yields $-2.96J_0$, which is presumably (see below) an *upper* bound for the ground state energy.

(ii) $\mu<1$: in this case, the average energy per spin is infinite. Since one may expect that all the strong bonds are satisfied, one should have, to leading order in N, $E=\Sigma_{i<j}|J_{ij}| = uN^{1/\mu}$, where u is distributed according to an asymmetrical Levy distribution of index μ [19]: this is precisely what is obtained using equations (3), (6). On

the other hand, this contribution to the free energy is independent of temperature and all the interesting effects are related to inter-cluster bonds which are, say, such that $|J|<1$, contributing to the free energy to order N. The question is thus if the remaining 'cluster-glass' retains some of the complexity of the SK spin-glass, or if the reduction of degrees of freedom is sufficient to allow a one-phase description. As was mentioned above, the result on the entropy and on the AT line shows that many phases are needed below a certain temperature. Analysis of the TAP equation however suggests that it might not be the case at zero temperature. Indeed, equation (8) reads, for $\mu<1$ and $T=0$, $m_i^{N+1}=m_0^{N+1}\,\mathrm{sgn}(J_{0i})$ if $|J_{0i}|>|\phi_i|$ and $m_0^{N+1}=\mathrm{sgn}\,[\Sigma_{|\phi_i|>|J_{0i}|}\;J_{0i}m_i^{N+1}+\Sigma_{|\phi_i|<|J_{0i}|}\,\mathrm{sgn}(J_{0i})\phi_i]$, where ϕ_i is the field acting on site i before the $(N+1)$th spin is added. The first equation ensures that the strongest bonds are satisfied, while the second sets (apparently correctly) the clusters' relative orientations. (The term in brackets can be seen as the effective field acting on the cluster containing site 0.) It may thus well be that 'replica symmetry' is restored at zero temperature.

In conclusion it appears that 'replica symmetry breaking' is needed for all $0<\mu<2$, at least for $0<T<T_{\mathrm{AT}}<T_{\mathrm{c}}$. Assuming an ultrametric distribution of states, one may write, following [1], the self-consistent equations for the generalized order parameter function $q(x)$ and the free-energy. (x is, as usual [1], a parameter between 0 and 1 indexing the level in the ultrametric tree.) We are currently working to solve these equations numerically, and will report on the results in a longer publication [20]. Preliminary analysis at $T=0$ reveals that:

(*a*) the ground state energy can only *decrease* when one allows for many states, at least at the 'one-step' level. This is at variance with the SK model, where the ground state energy increases when the replica symmetry is broken. Note however that no variational principle is available in our case to determine the location of the 'break point' x.

(*b*) the minimum overlap between two states is non-zero for $T=0$: this is also at variance with the SK model (1) and is again clearly related to the fact that the strong bonds impose a common 'skeleton' to all ground states.

An important outcome of the many-phase computation is the evolution of the phase space structure with temperature. The position of the breakpoint $x(T)$ (defined e.g. as the point where $\mathrm{d}q/\mathrm{d}x$ is maximum) is of crucial importance for the dynamics: the characteristic width of the distribution of energy barriers is given by $\Delta E=T/x(T)$ [1, 26]. In both the SK and the REM model [1, 27], $x(T)$ is proportional to T at low temperatures, that is $\Delta E\to$ constant. A decreasing $x(T)$ is needed to explain the anomalous dynamics found in [18] (see in particular figure 4 of [18], where x is called y [26]). The non-monotonic behaviour found above for the entropy could signal such an unexpected shape of $x(T)$. Further work is needed to answer this question.

In summary, we have presented the 'one-phase' (replica symmetric) solution of a spin-glass model with strongly fluctuating bonds, which we think is more appropriate to describe real spin-glasses than the usual Gaussian (SK) model. We have found that the spin-glass transition resembles a percolation transition, with a fraction of 'fast' spins decaying from 1 as the temperature is lowered. A second transition occurs for a lower temperature, towards a 'many-phase' spin-glass, presumably characterized by slow dynamics and aging. The degree of 'replica symmetry breaking', measured by the absolute value of the (negative) entropy, is found to be smaller than in the SK model, and non-monotonic with temperature (vanishing for $T=0$). The ultrametric 'many-phase' solution will be described in a future publication.

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